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THE EVOLUTION OF HOMOGENEOUS TURBULENCE IN A DENSITY-STRATIFIED MEDIUM. 3. ANALYSIS OF THE NEAR REGION

V. A. Babenko

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Development of a turbulent perturbation in a homogeneous flow moving within a medium that is linearly density-stratified across the direction of flow motion is described. The initial region of the evolution of turbulence with a large turbulent Reynolds number R_λ is considered.

We consider the evolution of turbulence created by a turbulizing grid in a medium with a homogeneous velocity field in the presence of a constant transverse density gradient $d\bar{\rho}/dx_2$ = const induced by the field of the force of gravity. In the analysis, we use the theory of quasihomogeneous turbulence developed in [1, 2].

It has been shown in [2, 3] that the model employed describes the emergence and propagation of internal gravitational waves that change substantially the flow character and the decay rate of the kinetic energy of the turbulence. In [2], the system of equations from [1] has been solved numerically, and in [3, 4] an analytical investigation of the same system has been carried out by the method of the small parameter ($\varepsilon = \mathrm{Fr}^2$), which was applicable everywhere except for, probably, a small initial portion at which the condition $t = \varepsilon T_\rho << 1$ is satisfied, where T_ρ is the time scale of the density field. The variable t has the meaning of time and is used in [3, 4] along with another time variable $\tilde{\tau} = \mathrm{Fr}\cdot\tau$ for a two-scale expansion in terms of the inverse Fr number, which is the ratio of the buoyancy and inertial forces. In [4], the final stage of degeneracy of the turbulence energy is investigated when $t \to \infty$, $R_\lambda \to 0$, and $\varepsilon << 1$. In what follows, we investigate another limiting case when $R_\lambda >> 1$ $\varepsilon << 1$.

The system of equations from [2] to be solved can be written in dimensionless form as follows:

$$\frac{dR_{22}}{d\tau} = -2\left\{\frac{1}{3}\left[d + \frac{9}{2}(1-d)\right]\left(3\frac{R_{22}}{E} - 1\right) + \frac{1}{3} + \frac{4}{5}Q\frac{T_u}{E}\mathrm{Fr}^2\right\}\frac{E}{T_u},$$

$$\frac{dE}{d\tau} = -2\left[1 + Q\frac{T_u}{E}\mathrm{Fr}^2\right]\frac{E}{T_u},$$

$$\frac{dT_u}{d\tau} = (F_u^{**} - 2) - 2\left[1 - d\left(\frac{2\sigma}{1+\sigma}\right)\left(\sigma_\infty + \frac{3}{5}\right)\frac{1}{R_\infty}\frac{T_u}{T_\rho}\right]\mathrm{Fr}^2\frac{QT_u}{E},$$

$$\frac{dQ}{d\tau} = -\mathrm{Fr}^2\left[\frac{2}{3} - \frac{1}{\mathrm{Fr}^2}\frac{R_{22}}{\Theta} - d\left(\frac{R_{22}}{E} - \frac{1}{3}\right)\right]\Theta - \tag{1}$$

$$-\left[(1-d)\left(\frac{1}{3} + 10\frac{R_{22}T_u}{ET_\rho}\right) + 2d\left(\sigma_\infty + \frac{3}{5}\right)\frac{1}{R_\infty}\frac{T_u}{T_\rho}\right]\frac{Q}{T_u},$$

$$\frac{d\Theta}{d\tau} = -2\left[1 - Q\frac{T_\rho}{\Theta}\right]\frac{\Theta}{T_\rho},$$

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$$\frac{dT_\rho}{d\tau} = (F^{**}_{\rho 2} - 2) + F^{**}_{\rho 1}\frac{T_\rho}{T_u} - d\frac{4}{3}\left(1 - \frac{3}{5R_\infty}\right),$$

where functions F^{**}_u, $F^{**}_{\rho 1}$, and $F^{**}_{\rho 2}$ from [1] depend on the turbulent Reynolds number via the dimensionless parameter $d(\mathrm{R}^2_\lambda) = 1 - 2/\sqrt{1 + \delta_u/\mathrm{R}^2_\lambda}$ (the constant δ_u is determined in [1] as $\delta_u = 2800$), which characterizes the inertial character of turbulence ($d \to 0$ ($\mathrm{R}_\lambda >> 1$) in the case of strong turbulence, and $d \to 1$ ($\mathrm{R}_\lambda \to 0$) in the case of weak turbulence); $F^{**}_u = \frac{11}{3} - \frac{13}{15}d$; $F^{**}_{\rho 1} = \frac{5}{3}(1 - d)$; $F^{**}_{\rho 2} = 2 + \frac{4}{3}d$.

The functions $R_\infty(\sigma)$ and $\sigma_\infty(\sigma)$ entering into Eq. (1) are asymptotic values of the ratio of time scales R and the turbulent Prandtl number at $\tau \to \infty$ in the case of passive stratification (Fr = 0) taken from [5]:

$$\sigma_\infty = \frac{3(1-\sigma)}{10\sigma}\left[1 - \left(\frac{2\sigma}{1+\sigma}\right)^{3/2}\right]^{-1}, \tag{2}$$

$$R_\infty = \frac{1}{5\sigma}\left[1 - \left(\frac{2\sigma}{1+\sigma}\right)^{3/2} + \sigma^{3/2}\right]\left[1 - 2\left(\frac{2\sigma}{1+\sigma}\right)^{1/2} + \sigma^{1/2}\right]^{-1}. \tag{3}$$

In what follows, we consider within an approximation of weak stratification the initial region of turbulence evolution, which spans a range of approximately the three first periods of oscillations of the functions and takes place during the dimensionless time interval $\Delta(\tilde{\tau}) \simeq 10$ (see [3]). The variable $t = \varepsilon T_\rho$ increases on this interval by about 0.3. This means that even if the condition $t << 1$ is satisfied in the initial point, the condition $t \sim 1$ characteristic for the far region [3] holds at the end of the mentioned portion, i.e., an interchange of the near and far regions takes place. In order to obtain analytical solutions at the initial region we will additionally assume that the parameter $d(\mathrm{R}^2_\lambda)$ is small (strong turbulence).

In order to simplify system of equations (1), we will use the same dimensionless variables as in [3, 4] but with a dissimilarity regarding the function q that describes the transverse mass flow. We will use the following notation:

$$q_1 = eT_\rho Q/E, \quad \vartheta = \varepsilon\Theta/E, \quad K = R_{22}/E, \quad R = T_u/T_\rho, \quad t = \varepsilon T_\rho, \quad \tilde{\tau} = e\tau,$$

where $e = \varepsilon^{1/2}$. The function K presents a portion of the energy of transverse pulsations in the kinetic energy of turbulence, the function R presents the ratio of time scales of the velocity field and the scalar field, and the function ϑ presents the ratio of the potential energy of scalar fluctuations in the field of the force of gravity $\varepsilon\Theta$ to the kinetic energy of turbulence E. In terms of the notation introduced, system of equations (1) will read as follows:

$$t\frac{dK}{d\tau} = \varepsilon\frac{-7d'(K - 1/3)}{R} + 2\varepsilon e q_1 (K - 4/5),$$

$$t\frac{dR}{d\tau} = \varepsilon\frac{4}{5}d(1 - R/R_\infty) - 2\varepsilon e q_1(1 - \alpha_2 R)R,$$

$$t\frac{d\vartheta}{d\tau} = \varepsilon 2\left(\frac{1}{R} - 1\right)\vartheta + 2\varepsilon e q_1(1 + \vartheta), \tag{4}$$

$$et\frac{dq_1}{d\tau} = t^2 A + q_1\varepsilon e\left[\frac{1}{R}\left(2 - \frac{d'}{3}\right) - 10d'K - \alpha_1 + p\right] + 2\varepsilon^2 q_1^2,$$

$$\frac{dt}{d\tau} = \varepsilon p, \quad t\frac{dE}{d\tau} = -\varepsilon\frac{2E}{R} - 2\varepsilon e q_1 E,$$

where (see [31]) $\alpha_1(d, \sigma) = 2d(\sigma_\infty + 3/5)/R_\infty$; $\alpha_2(d, \sigma) = (\sigma/(1 + \sigma))\alpha_1$; $p = (4d/5R_\infty + 5d'/3R) > 0$; $d' = 1 - d$; $A = K + \vartheta[\, d(K - 1/3) - 2/3]$.

Instead of the function $q = \varepsilon T_p Q/E$ from [3], in [4] we introduced the function $q_1 = eT_p Q/E$. In the initial region, as has been noted, the expression $t = \varepsilon T_p$ entering into the definition of q can take small values (e.g., $t = 0.01$ at $\varepsilon = 10^{-3}$ and the characteristic value $T_{p0} = 10$). The terms with q are correspondingly small. To avoid disrupting the expansion in terms of ε by the small order of q, we introduced the variable q_1. As will be shown in what follows, this completely corresponds to an expansion in terms of the small parameter from [3] if we set $\hat{q} = 0$.

Inasmuch as the functions of system (4) are related to the kinetic energy of turbulence E only via dependence on the parameter d, it is convenient to write an equation for d, which is equivalent to an equation for R_λ:

$$t\frac{d\,(d)}{d\tau} = p_1\left[\varepsilon\,\frac{F_u^{**} - 4}{R} - 2\varepsilon e\,(2 - \alpha_2 R)\,q_1\right], \tag{5}$$

where p_1 is the logarithmic derivative of d with respect to R_λ^2,

$$p_1 = \frac{d\,(d)}{d\,(R_\lambda^2)}\,R_\lambda^2 = (1/(1 + \delta_u/R_\lambda^2)^{1/2} - 1)\,(1 + (1 + \delta_u/R_\lambda^2)^{1/2})^{-1} = -\frac{dd'}{(1 + d)}.$$

Let us expand the derivatives in Eqs. (4), (5) in terms of two "independent" variables t and $\tilde{\tau}$ (e.g., $\partial K/\partial\tau = \varepsilon p\partial K/\partial t + e\partial K/\partial\tilde{\tau}$) and present the sought functions in the following form

$$K = \hat{K}\,(t, \tilde{\tau}) + \varepsilon\tilde{K}\,(t, \tilde{\tau}) + O\,(\varepsilon^2),\quad R = \hat{R}\,(t, \tilde{\tau}) + \varepsilon\tilde{R}\,(t, \tilde{\tau}) + O\,(\varepsilon^2),$$

$$\vartheta = \hat{\vartheta}\,(t, \tilde{\tau}) + \varepsilon\tilde{\vartheta}\,(t, \tilde{\tau}) + O\,(\varepsilon^2),\quad d = \hat{d}\,(t, \tilde{\tau}) + \varepsilon\tilde{d}\,(t, \tilde{\tau}) + O\,(\varepsilon^2),$$

$$q_1 = \hat{q}_1\,(t, \tilde{\tau}) + e\tilde{q}_1\,(t, \tilde{\tau}) + O\,(e^2), \tag{6}$$

where all functions except for q_1 are expanded in terms of ε, and q_1 is expanded in terms of $e = \varepsilon^{1/2}$. A substantiation of this form of expansion is presented elsewhere [3].

In the far region considered earlier in [3], the first group of terms of expansions (6) (with hats) presented values of functions averaged over internal waves, and the second group of terms (with tildes) presented oscillations generated by these waves.

Upon substituting (6) into Eqs. (4)-(5) and gathering terms with equal powers of e, we obtain for the first two terms of (6) the following set of equations written from left to right according to the order of e powers in the expansions:

$$\frac{\partial\hat{K}}{\partial\tilde{\tau}} = 0,\qquad t\hat{p}\,\frac{d\hat{K}}{dt} = -7\,\frac{\hat{d}'\,(\hat{K} - 1/3)}{\hat{R}},\qquad t\,\frac{\partial\tilde{K}}{\partial\tilde{\tau}} = 2\hat{q}_1\,(\hat{K} - 4/5);$$

$$\frac{\partial\hat{R}}{\partial\tilde{\tau}} = 0,\qquad t\hat{p}\,\frac{d\hat{R}}{dt} = \frac{4}{5}\,\hat{d}\,(1 - \hat{R}/R_\infty),\qquad t\,\frac{\partial\tilde{R}}{\partial\tilde{\tau}} = -2\hat{q}_1\,(1 - \hat{\alpha}_2\hat{R})\,\hat{R};$$

$$\frac{\partial\hat{\vartheta}}{\partial\tilde{\tau}} = 0,\qquad t\hat{p}\,\frac{d\hat{\vartheta}}{dt} = 2\left(\frac{1}{\hat{R}} - 1\right)\hat{\vartheta},\qquad t\,\frac{\partial\tilde{\vartheta}}{\partial\tilde{\tau}} = 2\hat{q}_1\,(1 + \hat{\vartheta});$$

$$\frac{\partial\hat{d}}{\partial\tilde{\tau}} = 0,\qquad t\hat{p}\,\frac{d\,(\hat{d})}{dt} = \hat{p}_1\left[-\frac{\frac{1}{3} + \frac{13}{15}\hat{d}}{\hat{R}}\right],\qquad t\,\frac{\partial\tilde{d}}{\partial\tilde{\tau}} = -2p_1\hat{q}_1\,(2 - \hat{\alpha}_2\hat{R}); \tag{7}$$

$$\frac{\partial\hat{E}}{\partial\tilde{\tau}} = 0,\qquad t\hat{p}\,\frac{d\hat{E}}{dt} = -2\,\frac{\hat{E}}{\hat{R}},\qquad t\,\frac{\partial\tilde{E}}{\partial\tilde{\tau}} = -2\hat{q}_1\hat{E};$$

$$t^2\hat{A} = 0,\qquad \frac{\partial\hat{q}_1}{\partial\tilde{\tau}} = t\tilde{A},\qquad t\hat{p}\,\frac{\partial\hat{q}_1}{\partial t} + t\,\frac{\partial\tilde{q}_1}{\partial\tilde{\tau}} = \hat{q}_1\hat{b},$$

where $\tilde{A}$ is the second term of the expansion in terms of ε for $A(k, \vartheta, d)$, $A(t, \tilde{\tau}) = \hat{A}(t, \tilde{\tau}) + \varepsilon\tilde{A}(t, \tilde{\tau}) + \ldots$, and $\hat{b} = [\hat{R}^{-1}(2 - \hat{d}'/3) - 10\hat{d}'\hat{K} - \hat{\alpha}_1 + \hat{p}]$, the functions with hats depend only on the two first terms in (6). An auxiliary equation for the function A which can be obtained from (4) and (5), is as follows:

$$t\frac{dA}{d\tau} = 2\varepsilon c_1 + 2\varepsilon e c_2 q_1 ,$$

where

$$c_1 = -\frac{7}{2}(1 + \vartheta d)\, d'\left(K - \frac{1}{3}\right) R^{-1} + \vartheta\left[d\left(K - \frac{1}{3}\right) - \frac{2}{3}\right](R^{-1} - 1) + \vartheta\left(K - \frac{1}{3}\right)\frac{p_1}{2}(F_u^{**} - 4)\, R^{-1} ;$$

$$c_2 = (1 + \vartheta d)\left(K - \frac{4}{5}\right) + \left[d\left(K - \frac{1}{3}\right) - \frac{2}{3}\right](1 + \vartheta) + \vartheta\left(K - \frac{1}{3}\right) p_1 (2 - \alpha_2 R) .$$

Carrying out a similar expansion in the equation for A, we arrive at

$$\frac{\partial\hat{A}}{\partial\tilde{\tau}} = 0 , \quad t\hat{p}\frac{d\hat{A}}{dt} = 2\hat{c}_1 , \quad t\frac{\partial\tilde{A}}{\partial\tilde{\tau}} = 2\hat{q}_1\hat{c}_2 . \tag{8}$$

Let us analyze system (7), (8). Let us agree to index the rows of system (7) by an additional number (from one to six) and the columns of (7) and (8) by an additional letter (from a to c). As in [3], the first terms of expansions (6) for all functions except q_1 depend solely on the variable t (7.1a -7.5a), (8a). Let us differentiate (7.6b) with respect to $\tilde{\tau}$ and substitute into (8c). We obtain a wave equation for $\hat{q}_1$:

$$\frac{\partial^2\hat{q}_1}{\partial\tilde{\tau}^2} + \omega^2\hat{q}_1 = 0 , \tag{9}$$

where

$$\omega^2(t) = -2\hat{c}_2(t) . \tag{10}$$

Expression (10) for frequency coincides in its form with that from [3] with the sole exception that here oscillations with frequency ω take place in the first term of expansion $\hat{q}_1$, and not in the second one $\tilde{q}$, as in [3]. The first term $\hat{q}$ in the expansion for q from [3] was determined from the relationship (7.6a) and corresponded to a nonzero averaged (over vibrations) turbulent mass flow q.

In system (7.1b)-(7.5b), (7.6a) for determination of $\hat{K}$, $\hat{d}$, $\hat{\vartheta}$, $\hat{E}$, and $\hat{R}$, one of the equations is redundant. At small t, the equation $t^2\hat{A} = 0$ should be discarded, since in this case it does not follow from (7.6a) that $\hat{A} = 0$. On the contrary, if $t^2 \gg \varepsilon$, then one of Eqs. (7.1b)-(7.5b) should be discarded as a condition of a lower order compared to (7.6a). It is evident that this is Eq. (7.3b), since the function $\hat{\vartheta}$ enters only into two relationships: (7.6a) and (7.3b).

In [3], with $\hat{q}(t)$ known, expression (7.6c) was used to define $\tilde{q}$. Here, it is used to find the dependence $\hat{q}_1(t)$, and the derivative $t\partial\tilde{q}_1/\partial\tilde{\tau}$ in (7.6c) should be neglected to close the analysis. On doing this, the variables in Eqs. (7.6b) and (7.6c) are separated by the substitution $\hat{q}_1(t, \tilde{\tau}) = \hat{q}_1'(t)\hat{q}_1''(\tilde{\tau})$, and one can write

$$\frac{d^2\hat{q}_1''}{d\tilde{\tau}^2} + \omega^2\hat{q}_1'' = 0 , \quad t\hat{p}\frac{d\hat{q}_1'}{dt} = \hat{q}_1'\hat{b} . \tag{11}$$

The first of relation (11) serves for evaluation of the phase of oscillations from the variable $\tilde{\tau}$, and the second serves for evaluation of the amplitude depending on the variable t.

It should be noted that Eqs. (7.1b)-(7.5b), (11) could be obtained as a degenerate case of the analysis [3], if we set there $\hat{q} = 0$. In the near region, in the absence of a smooth component $\hat{q}(t)$, the second term in the approximate solution [3] $\tilde{q}(t, \tau)$ plays the role of the first one, and relations (11) hold for this term. Thus, it is shown that the approximation of the near region is a particular case of the more general analysis [3]. In what follows, we will compare dependences obtained with results from [3].

Equations for $\hat{K}$, $\hat{\vartheta}$, and $\hat{E}$ in system (7.1b)-(7.5b) can be solved by solving the pair of equations for $\hat{d}$ and $\hat{R}$, and in the expression for $\hat{R}$ we can introduce $\hat{d}$ as a new independent variable. In this case the variables in (7.2b) are separated

$$\frac{d\hat{R}}{\hat{R}(1-\hat{R}/R_\infty)} = \frac{4}{5}\,\frac{(1+\hat{d})\,d(\hat{d})}{(1-\hat{d})\left(\frac{1}{3}+\frac{13}{15}\hat{d}\right)}.$$

By integrating the given equation from the initial state (with the index 0) to the current one, we obtain

$$\ln\frac{\hat{R}(1-R_0/R_\infty)}{R_0(1-R/R_\infty)} = \ln\left[\left(\frac{5+13\hat{d}}{5+13d_0}\right)^{4/13}\frac{1-d_0}{1-\hat{d}}\right]^{4/3},$$

from which the expression for $\hat{R}^{-1}$ follows:

$$\frac{1}{\hat{R}} = \frac{1}{R_\infty} + \left(\frac{1}{R_0}-\frac{1}{R_\infty}\right)\left[\left(\frac{5+13d_0}{5+13\hat{d}}\right)^{4/13}\frac{1-\hat{d}}{1-d_0}\right]^{4/3}. \tag{12}$$

Solutions in the form $\hat{K}(\hat{d})$ and $\hat{E}(\hat{d})$ can be obtained for Eqs. (7.1b) and (7.5b). We combine (7.1b) and (7.5b) with (7.4b) and integrate:

$$\hat{K} = \frac{1}{3} + \left(K_0-\frac{1}{3}\right)\left(\frac{\hat{d}}{d_0}\right)^{-21}\left(\frac{5+13\hat{d}}{5+13d_0}\right)^{168/13}, \tag{13}$$

$$\hat{E} = E_0\left(\frac{1-\hat{d}}{1-d_0}\right)^{10/3}\left(\frac{d_0}{\hat{d}}\right)^{6}\left(\frac{5+13\hat{d}}{5+13d_0}\right)^{8/3}. \tag{14}$$

It is also easy to express in quadratures the dependences of other functions in (7) on $\hat{d}$. Thus, for $\hat{\vartheta}(\hat{d})$ we have

$$\hat{\vartheta} = \vartheta_0\exp\left(\int_{d_0}^{\hat{d}}\frac{2(1+\hat{d})(1-\hat{R})\,d(\hat{d})}{\hat{d}\left(\frac{1}{3}+\frac{13}{15}\hat{d}\right)(1-\hat{d})}\right), \tag{15}$$

where $1-\hat{R}$ is calculated as a function of $\hat{d}$ according to (12). It is more convenient to take the integral in (15) numerically, as in the expression for $\hat{d}(t)$ emerging upon performing integration in (7.4b):

$$\int_{d_0}^{\hat{d}}\frac{(1+\hat{d})\,\hat{p}\hat{R}\,d(\hat{d})}{\hat{d}\left(\frac{1}{3}+\frac{13}{15}\hat{d}\right)(1-\hat{d})} = \int_{t_0}^{t}\frac{dt}{t}. \tag{16}$$

By defining functions $\hat{d}(t)$, $\hat{R}(\hat{d}(t))$, $\hat{K}(\hat{d}(t))$, and $\hat{\vartheta}(\hat{d}(t))$, equation for the amplitude (11b) can also be solved in quadratures. However, it is more simple to find the amplitude from the numerical solution of system (7.1b)-(7.4b), (11b).

Expressions (12)-(14) make it possible to carry out an analysis. Thus, in the near region the change of variables $\hat{K}$, $\hat{E}$, and $\hat{R}$ is governed by the change of the sole parameter $\hat{d}$, i.e., the turbulent Reynolds number. The molecular Prandtl number does not affect the dependences $\hat{K}(\hat{d})$ and $\hat{E}(\hat{d})$. This also pertains to the dependence of the ratio $(\hat{R}^{-1} - R_\infty^{-1})/(R_0^{-1} - R_\infty^{-1})$ on $\hat{d}$, which is universal in the near region. Since the ratio $\hat{d}/d_0$ grows, relation (13) describes the convergence of $\hat{K}$ on the isotropic value of 1/3. The value in the square bracket in (12) is positive. Therefore, an increase or decrease in $\hat{R}$ depends on the sign of the difference $R_0^{-1} - R_\infty^{-1}$. If this difference is negative, $\hat{R}$ drops; when the difference is positive, the quantity $\hat{R}$ increases. The rate of change in $\hat{R}$ decreases with time. This is determined by the presence of the first, decreasing, factor in the quare bracket in (12).

A more complete analysis can be carried out with the use of the assumption on the small value of the parameter $\hat{d}$, $\hat{d} \ll 1$, in the near region. In this case Eqs. (7.1b)-(7.5b), (11b) can be simplified:

$$\frac{5}{3}t\frac{\hat{d}(K - 1/3)}{dt} = -7(\hat{K} - 1/3), \quad \frac{5}{3}t\frac{d\hat{R}}{dt} = \frac{4}{5}\hat{d}(1 - \hat{R}/R_\infty)\hat{R},$$
$$\frac{5}{3}t\frac{d\hat{\vartheta}}{dt} = 2(1 - \hat{R})\hat{\vartheta}, \quad \frac{5}{3}t\frac{d(\hat{d})}{dt} = \frac{1}{3}\hat{d},$$
$$\frac{5}{3}t\frac{d\hat{E}}{dt} = -2\hat{E}, \quad \frac{5}{3}t\frac{d\hat{q}_1'}{dt} = 10\hat{q}_1'\left(\frac{1}{3} - \hat{K}\hat{R}\right). \tag{17}$$

By introducing the notation $y = t/t_0$, we write solutions of the first, second, fourth, and fifth equations in (17):

$$\hat{d}/d_0 = y^{1/5}, \quad (\hat{K} - 1/3)/(K_0 - 1/3) = (\hat{d}/d_0)^{-21} = y^{-21/5},$$
$$\hat{E}/E_0 = (\hat{d}/d_0)^{-6} = y^{-6/5},$$
$$\left(\frac{1}{R} - \frac{1}{R_\infty}\right) \Big/ \left(\frac{1}{R_0} - \frac{1.}{R_\infty}\right) = \exp\left(-\frac{12}{5}(d - d_0)\right) = \exp\left(-\frac{12}{5}d_0(y^{1/5} - 1)\right). \tag{18}$$

We solve the equation for $\hat{\vartheta}$ from (17) approximately. At a small d_0, the argument of the exponential in (18) is small, and therefore the function $\hat{R}(y)$ changes slowly, which gives grounds to consider the "frozen" value $\hat{R}$ in the equation for $\hat{\vartheta}$. This assumption leads to the power solution

$$\hat{\vartheta} = \vartheta_0 y^{\frac{6}{5}(1-\hat{R})}. \tag{19}$$

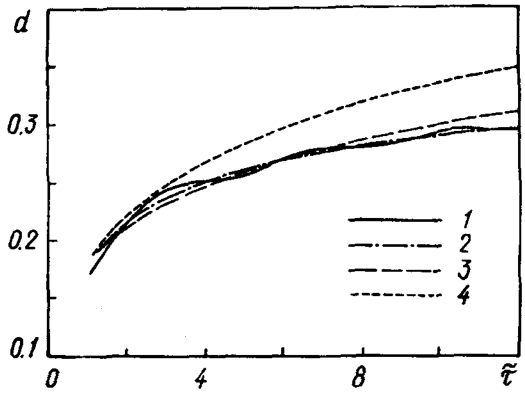
For $\hat{R} > 1$, expression (19) determines the power decrease in $\hat{\vartheta}$, and, on the contrary, a power growth for $\hat{R} < 1$. In accordance with the above-presented, expression (19) holds only within a very narrow initial region $t^2 \ll 1$. At larger values of t, it should be replaced by an algebraic relation that following from the condition $\hat{A} = 0$, namely:

$$\hat{\vartheta} = \frac{-\hat{K}}{\hat{d}(\hat{K} - 1/3) - 2/3}. \tag{20}$$

Let us consider qualitatively equation (11b) for $\hat{q}_1'$. Taking into account the fast convergence of $\hat{K}$ on 1/3 (18) and considering the "frozen" value of $\hat{R}$, we simplify (11b) to obtain

$$\frac{t\,d\hat{q}_1'}{\hat{q}_1'\,dt} \simeq 2(1 - \hat{R}). \tag{21}$$

According to (21), the amplitude of oscillations of the mass flow follows a law close to a power function; it decreases at $\hat{R} > 1$ and increases for $\hat{R} < 1$.



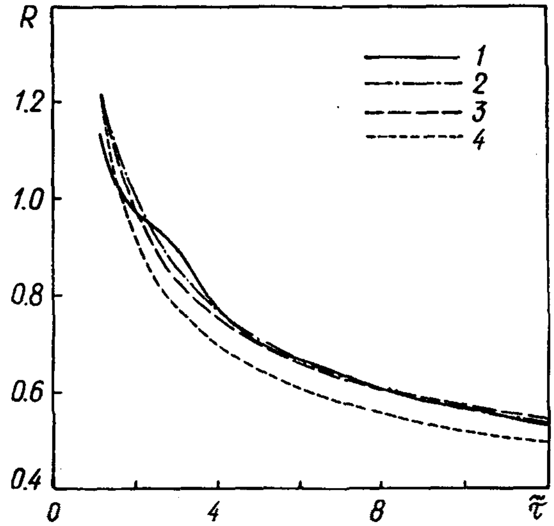


Fig. 1. Comparison of dependences for functions d and $\hat{d}$: 1) calculation for d by system (1); 2) $\hat{d}$ by formulas from [3]; 3) $\hat{d}$ by (16); 4) $\hat{d}$ by (18).

Fig. 2. Comparison of dependences for functions R and $\hat{R}$: 1) calculation for R by system (1); 2) $\hat{R}$ by formulas from [3]; 3) $\hat{R}$ by (12); 4) $\hat{R}$ by (18).

Let us now consider system (7.1c)-(7.5c), (11a), which describes the dependence of the functions on $\tilde{\tau}$. The solution of Eq. (9) is as follows:

$$\hat{q}_1 = \hat{q}_1'(t_1)\sin(\omega\tilde{\tau} + \varphi_0). \tag{22}$$

For other functions of system (7.1c)-(7.5c) the dependences follow laws similar to those obtained in [3] for the far region:

$$\tilde{K}' = -2\frac{\hat{q}_1'(\hat{K} - 4/5)}{t\omega}, \quad \tilde{K}'' = \cos(\omega\tilde{\tau} + \varphi_0); \tag{23}$$

$$\tilde{R}' = 2\frac{\hat{q}_1'(1 - \hat{\alpha}_2\hat{R})\hat{R}}{t\omega}, \quad \tilde{R}'' = \cos(\omega\tilde{\tau} + \varphi_0); \tag{24}$$

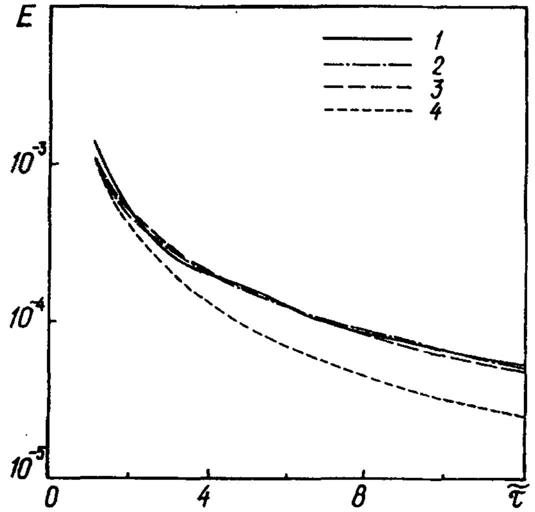
$$\tilde{\vartheta}' = -2\frac{\hat{q}_1'(1 + \hat{\vartheta})}{t\omega}, \quad \tilde{\vartheta}'' = \cos(\omega\tilde{\tau} + \varphi_0); \tag{25}$$

$$\tilde{d}' = 2\hat{p}_1\frac{\hat{q}_1'(2 - \hat{\alpha}_2\hat{R})}{t\omega}, \quad \tilde{d}'' = \cos(\omega\tilde{\tau} + \varphi_0); \tag{26}$$

$$\tilde{E}' = 2\frac{\hat{q}_1'\hat{E}}{t\omega}, \quad \tilde{E}'' = \cos(\omega\tilde{\tau} + \varphi_0). \tag{27}$$

At small $\hat{d}$, the expression for the frequency of internal waves $\omega^2 = -2\hat{c}_2$ can be simplified in the following manner:

$$\omega^2 = -2\left[(\hat{K} - 4/5) - \frac{2}{3}(1 + \hat{\vartheta})\right]$$



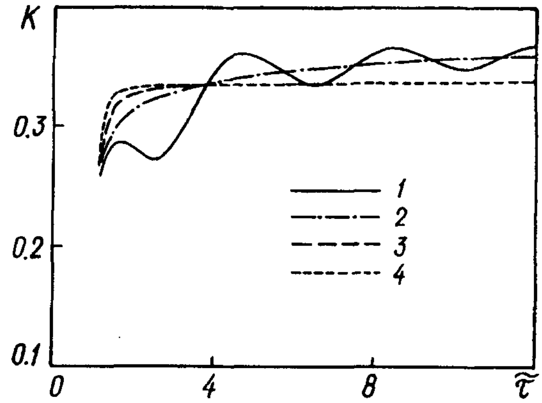


Fig. 3. Comparison of dependences for functions E and $\hat{E}$: 1) calculation for E by system (1); 2) $\hat{E}$ by formulas from [3]; 3) $\hat{E}$ by (14); 4) $\hat{E}$ by (18).

Fig. 4. Comparison of dependences for functions K and $\hat{K}$: 1) calculation for K by system (1); 2) $\hat{K}$ by formulas from [3]; 3) $\hat{K}$ by (13); 4) $\hat{K}$ by (18).

and, by substituting then $\hat{K}$ from (18) and $\hat{\vartheta}$ from (19), one can write

$$\omega^2 = 2\frac{4}{15} - 2\left[(K_0 - 1/3)\, y^{-21/5} - \vartheta_0\, y^{\frac{6}{5}(1-\hat{R})}\right]. \tag{28}$$

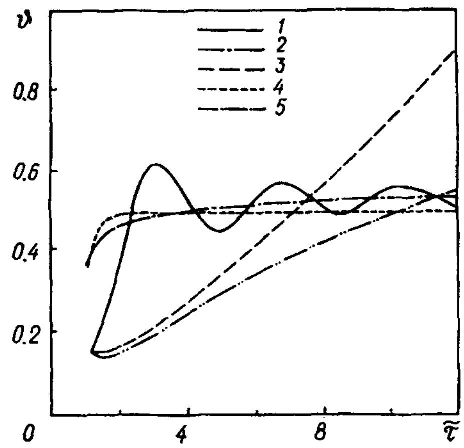
In order to verify the obtained analytical and semianalytical dependences we compared the results with a numerical solution of original system of differential equations (1). Initial data for the numerical computation corresponded to the following set of initial values from [2] coinciding with the set of experimental points from [6] for water with $\sigma = 800$ and $\mathrm{Fr} = 3.67\cdot 10^{-2}$: $R_{22_0} = 3.47\cdot 10^{-4}$, $E_0 = 1.33\cdot 10^{-3}$, $T_{\rho_0} = 38.4$, $T_{u_0} = 43.3$, $Q_0 = 3.31\cdot 10^{-3}$, and $\Theta_0 = 0.153$.

Figure 1 presents a comparison of analytical expressions for the function $d(\tilde{\tau})$ in the near region (16) and (18) with the numerical solution. In the numerical solution, the function $d(\hat{\tau})$ grows somewhat slower than in the analytical solution (18), which sets up the "one-fifth" law. More general expression (16) corresponds better to the numerical solution. Here and in what follows, initial conditions for the functions with hats differ from those presented in the preceding paragraph. They were determined as described elsewhere [3].

A similar comparison of the complete numerical solution for $R(\tilde{\tau})$, a smoothed solution in the far region (Eqs. (9), (15) from [3]), and two analytical expressions in the near region of the general form of (12) for $d \ll 1$ (18) is presented in Fig. 2. It is evident that the generally good agreement between the numerical solution and analytical expressions worsen with time.

In Fig. 3, we compared the kinetic energy of turbulence obtained from expression (14), the solution of complete system (1), and the solution of the "smoothed" system from [3] in the far region with analytical expression (18). Here, as in the preceding figures, the exact solution (14) is close to the complete numerical solution, whereas analytical power solution (18), which is characteristic for turbulence propagation in an isotropic medium, provides underestimated values of the kinetic energy.

The analytical expressions for the function $\hat{K}$ at a small parameter d (18) and in the general case (13) correspond less to the numerical solution of the complete system, since their feature is the presence of the asymptotic $\hat{K} = 1/3$ (Fig. 3). The pronounced intermediate asymptotic $\hat{K} = 1/3$ is observed in the numerical solution of the complete system only at very small ε. The slope of the flat portion in the dependence $\hat{K}(\tilde{\tau})$ becomes steeper with ε, and the asymptotic is lost. The solution of the "smoothed" solution corresponds well to the oscillation-



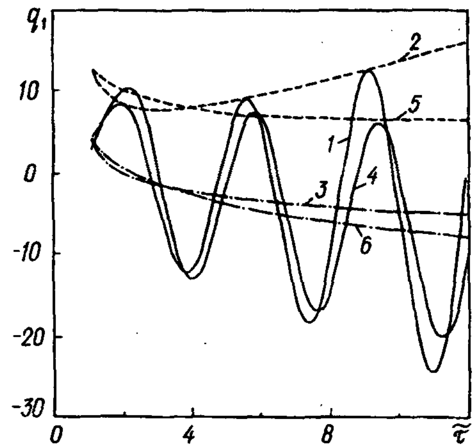


Fig. 5. Comparison of various dependences for the function ϑ: 1) calculation for ϑ by system (1); 2) $\hat{\vartheta}$ by formulas from [3]; 3) $\hat{\vartheta}$ by (15); 4) $\hat{\vartheta}$ by (20).

Fig. 6. Comparison of various dependences for the function q_1 in the initial region.

averaged complete numerical solution. Recall that the smoothing in [3] is carried out by replacing the differential equation for q in (1) by the approximate algebraic equation

$$\hat{q} = -\hat{c}_1/\hat{c}_2, \tag{29}$$

which is obtained by differentiating condition (7.6a) $t^2\hat{A} = 0$ with respect to t.

The time interval on which a correspondence is observed for $\hat{\vartheta}$ between the solution of complete system (1) and two approximate solutions – the power solution in the form (19) and the quadrature solution (15) – appears to be extremely narrow (Fig. 5). The condition $t \gg \varepsilon$ took place in the given initial data, and, in accordance with the above-discussed, an approximation for $\hat{\vartheta}$ more exact than (15) and (11) can be found from relation (20), which is similar to expression (9) from [3]. Figure 5 presents the dependence obtained in this manner where $\hat{d}$ and $\hat{K}$ are calculated by equations of system (7).

In system (7.1b)-(7.5b) and (11) for determination of first terms of expansions (6), the mass flow $\hat{q}_1$ being formed is a kind of passive admixture, i.e., it is unambiguously determined upon calculating all other functions. Physically, the near region corresponds to the initial evolution stage of stratified turbulence, when stratification affects only formations of the largest scale. According to the approximate solutions obtained, the effect of stratification in the near region manifests itself as oscillations superimposed on the smooth variation of functions $\hat{K}$, $\hat{R}$, $\hat{\vartheta}$, $\hat{E}$, and $\hat{d}$, which is not influenced by this effect and depends solely on reduction of the turbulent Reynolds number.

The use of this approximation is justified by the fact that this makes it possible to obtain analytical solutions. On the other hand, it leads (see Figs. 1-5) to less accurate dependences than more general ones from [3]. According to the expressions for turbulent mass flow (11), the flow q_1 is presented in the form of a harmonic function with a varying amplitude without an additive component, which contradicts the numerical calculation (see Fig. 6). This takes place due to the fact that a zeroth-order equation in expansion (6) for q_1 (7.6a) was used to determine the function $\hat{\vartheta}$, and Eq. (7.4b) was discarded as contradictory. Thus, one equation and one additional function, namely, the additive component of the transverse mass flow, disappeared from system (7).

To overcome this shortcoming, one can, having determined approximate solutions for all other functions, pass to the variable $q = \mathrm{Fr}\cdot q_1$ in the fourth equation (4) and, by carrying out expansion according to the same scheme (6), obtain a more exact representation for q_1, where the additive component

$$q_1 = -\hat{c}_1/e\hat{c}_2 + \hat{q}_1' \sin(\omega\tilde{\tau} + \varphi_0). \tag{30}$$

is already present.

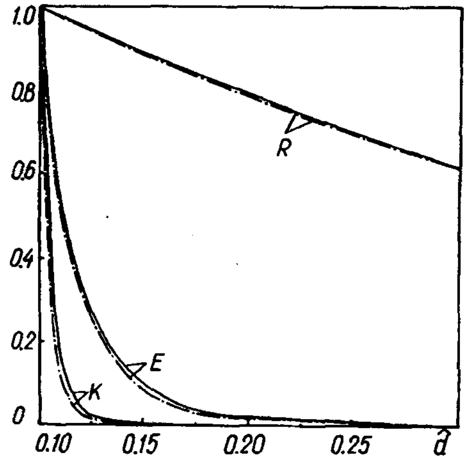


Fig. 7. Comparison of power dependences (12)-(14) with asymptotic formulas (18) for corresponding functions in the case $d \ll 1$.

In Fig. 6, Eq. (30) is compared with the numerical solution of (1) (solid curves 1 and 4, respectively). The same figure presents also an algebraic dependence for the averaged mass flow $q_1 = -\hat{c}_1/e\hat{c}_2$, where coefficients $\hat{c}_1$ and $\hat{c}_2$ are calculated by exact dependences from [3] (curve 6) and by approximate ones (12), (13), (16), and (20) (curve 3). As a whole, the approximate solution coincides satisfactorily with the numerical one, and the maximum discrepancy is observed for amplitudes (amplitude envelopes are presented by curves 2 for 1 and 5 for 4).

Summing up the comparative analysis presented, we notice that the approximation of the near region describes satisfactorily the true dependences for functions R, d, E, and K within the entire initial region, including an area outside the boundary of the region. At the same time, the use of an approximation of a large turbulent Reynolds number ($d = 0$) leads to a large error. It results mainly from noncoincidence of the actual dependence for the parameter $d(t)$ with the asymptotic law of "one fifth" (18), since the dependences of ratios $(\hat{K} - 1/3)/(K_0 - 1/3)$, $\hat{E}/E_0$, and $(1/R - 1/R_\infty)/(1/R_0 - 1/R_\infty)$ (Fig. 7, from top to bottom) on $\hat{d}$ constructed by formulas (12)-(14) and (18) differ insignificantly within the considered interval of variation of the parameter $\hat{d}$ (Fig. 7).

For the functions $\hat{q}_1(\tilde{\tau})$ and $\hat{\vartheta}(\hat{\tau})$, algebraic relations in the far region, i.e., (20) and (29), work for the presented set of initial data immediately from the initial conditions. In fact, the initial conditions set up the amplitude and phase of oscillations of these functions, and their mean values are determined from algebraic expressions in which initial conditions are not present.

Inasmuch as modeling of the interaction of turbulence internal waves and true turbulence is crucially important for description of stratified flows, information on which terms and the extent to which they are responsible for the generation of waves, their amplitude, frequency, and decay laws of individual characteristics can be useful when modeling stratified turbulence and for comparing individual terms of various models. The fact that this work considers only one turbulence model [1, 2] does not actually minimize the generality of the analysis. A review of modern second-order moment models suitable for description of stratified turbulence [8] shows that in the best models, the structure of the equations is very close to the structure of system (1), and a series of algebraic relations is proposed for description of turbulent transverse mass flow. At small inverse Frude numbers, the analysis carried out in [3, 4] and in the present work can be repeated almost verbatim. The mathematical investigation by means of the method of the small parameter carried out here and in [3] demonstrates the role of algebraic relations (in the given case, q and ϑ) in introducing averaging over oscillations.

NOTATION

τ^*, dimensional time; U, flow rate; M, size of the lattice cell; $\tau = \tau^* U/M$, dimensionless time; $E^* = (\overline{u_1^2} + \overline{u_2^2} + \overline{u_3^2})$, doubled kinetic energy of turbulence; $E = E^*/U^2$, dimensionless kinetic energy; $R_{22} = \overline{u_2^2}/U^2$, vertical component of the tensor of velocity pulsations; ε_ρ, dissipation rate of density pulsations; ε_u, dissipation rate of velocity pulsations; $T_u = (E^* U)/(\varepsilon_u M)$, time scale of velocity field; $T_\rho = (\overline{\rho^2} U)/(\varepsilon_\rho M)$, time scale of density field; $R = T_u/T_\rho$; $Q = (-\overline{u_2 \rho})/(UM d\overline{\rho}/dx_2)$, dimensionless turbulent transverse mass flow; $\Theta = \overline{\rho^2}/(M d\overline{\rho}/dx_2)^2$, squared velocity pulsations; σ, molecular Prandtl number; $\varepsilon = \mathrm{Fr}^2$, small parameter; $\mathrm{Fr} = N_{\mathrm{BV}} M/U$, Froude number; $N_{\mathrm{BV}} = ((g/\overline{\rho})(d\overline{\rho}/dx_2))^{1/2}$, Brunt–Väisälä number; $\mathrm{R}_\lambda = (5ET_u \mathrm{Re})^{1/2}$, turbulent Reynolds number; $\mathrm{Re} = UM/\nu$, Reynolds number; $\tilde{\tau} = \varepsilon^{1/2}\tau$; $t = \varepsilon T_\rho$; $e = \sqrt{\varepsilon}$.

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